U(1) $\times$ SU (2) GAUGE THEORY OF UNDERDOPED HIGH T_c CUPRATES VIA CHERN–SIMONS BOSONIZATION

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Abstract

We outline the basic ideas involved in a recently proposed [17] derivation of a gauge theory for underdoped cuprates in the "spin–gap phase", performed essentially step by step starting from the $t-J$ model, considered as a model Hamiltonian for the CuO_2 layers. The basic tool is the $U(1) \times SU(2)$ Chern–Simons bosonization, to which it is dedicated a somewhat detailed discussion. The basic output is a "spin–gap" not vanishing in any direction and an antiferromagnetic correlation length proportional to the inverse square root of doping concentration, in agreement with data deduced from the neutron experiments. The model also exhibits a small half–pocket Fermi surface around $(\pm\pi/2, \pm\pi/2)$ and a linear in temperature dependence of in–plane resistivity in certain temperature range.

1. High T_c Cuprates and the t–J model

A common structural feature of high T_c cuprates is the presence of electronically active CuO_2 layers, alternating with (insulating) block layers along the crystalline c–axis. In the CuO_2 layers of undoped materials the $3d$ shell of copper has a hole (primarily in the highest energy $3d_{x^2-y^2}$ orbital) while the $2p$ shell of the oxygen is filled. The spin $\frac{1}{2}$ moments of the Cu are antiferromagnetically ordered at low temperatures and a strong on–site Coulomb repulsion acts in the $3d$ orbital, inhibiting double occupation. As the materials are doped, holes (or electrons) are introduced in the CuO_2 layers.

In terms of doping concentration (δ) and temperature (T), a "typical" phase diagram is drawn in fig.1 (patterned on $La_{2-\delta}Sr_\delta CuO_4$ compounds). It exhibits an antiferromagnetic (AF) insulating phase near $\delta = 0$ (for sufficiently low temperatures), a superconducting (SC) phase for an intermediate doping (e.g. $0.1 \lesssim \delta \lesssim 0.2$ for $LaSrCuO$ compounds). The materials with doping concentration exhibiting highest T_c are called optimally doped; cuprates with lower or higher doping concentration are called underdoped or overdoped, respectively. At optimal doping, in the phase diagram above the SC region there is a region characterized by an anomalous metallic behaviour (e.g. linear in T in–plane resistivity ρ_{ab} [1]; anomalous spin lattice relaxation rate $\frac{1}{T_1 T} \sim \frac{1}{T^\alpha}, \alpha \sim 1$ [2]; large 2D Fermi surface whose volume is consistent with the Luttinger theorem [3]). Moving towards the underdoped region there is a crossover to the so–called "spin–gap phase", exhibiting distinctive phenomena (e.g. a minimum of $\rho_{ab}(T)$ for small enough δ [4]; a maximum in $\frac{1}{T_1 T}$ at low T [5]; in some materials small half–pocket like 2D Fermi surface around $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$[6]). In the overdoped region the materials appear to show an essentially "normal" metallic behaviour.

In this paper we will be interested in the underdoped (spin–gap) region. A model Hamiltonian for the CuO_2 planes (at low doping) has been proposed by Zhang and Rice [7], following a suggestion by Anderson [8], roughly on the basis of the following considerations (see [9] for a more precise and detailed discussion). The holes introduced by doping go primarily into symmetrized O–orbitals around the Cu ion and they form a spin singlet with the spin moment of copper (see fig. 2). A spin singlet (Cu–hole /O–hole) in one CuO_4 has a relevant nearest neighbour (n.n.) hopping, since each CuO_4 has an O–site in common with the n.n. CuO_4. The low energy physics is then believed to be dominated by the motion of these spin–singlets in the AF background of Cu spin moments. The Hamiltonian proposed for the system is given by

$$H=P_G[\sum_{<ij>}\sum_{\alpha}-t(c^{\dagger}_{i\alpha}c_{j\alpha}+h.c.)+J\vec{S}_i\cdot\vec{S}_j]P_G, \tag{1.1}$$

where i runs over the sites of the (square) lattice defined by the position of the Cu ions, the sum over α runs over spin indices (spin up=1, spin down=2). In eq. (1.1), $c_{i\alpha}$ denotes a spin $\frac{1}{2}$ fermion operator, P_G is the Gutzwiller projection, eliminating double occupation, modelling on–site Coulomb repulsion and the second term is an AF– Heisenberg Hamiltonian, where the spin $\vec{S}_i$ is given by

$$\vec{S}_i=c^{\dagger}_{i\alpha}\frac{\vec{\sigma}_{\alpha\beta}}{2}c_{i\beta}. \tag{1.2}$$

The system described by the Hamiltonian (1.1) is called "$t-J$ model".
According to [9], to match with the physics of high T_c cuprates one should take $\frac{J}{t}\sim\frac{1}{3}$.
We analyse the $t-J$ model with a path–integral approach (see e.g. [10]). We write the euclidean action, corresponding to the Hamiltonian (1.1), with chemical potential μ, at temperature T, in terms of spin $\frac{1}{2}$ (Grassmann) fermionic fields $\Psi_\alpha,\Psi^*_\alpha$. The action is given by

$$\begin{aligned}S(\Psi,\Psi^*)=\int_0^\beta dx^0\sum_i\Psi^*_{i\alpha}(\partial_0+\delta)\Psi_{i\alpha}+\sum_{<ij>}\{-t(\Psi^*_{i\alpha}\Psi_{j\alpha}+h.c.)\\ -\frac{J}{2}|\Psi^*_{i\alpha}\Psi_{j\alpha}|^2\}+\sum_{i,j}u_{i,j}\Psi^*_{i\alpha}\Psi^*_{j\beta}\Psi_{j\beta}\Psi_{i\alpha},\end{aligned} \tag{1.3}$$

where the two–body potential $u_{i,j}$ is given by

$$u_{i,j}=\begin{cases}+\infty & i=j\\ -\frac{J}{4} & i,j\ \text{ n.n}\end{cases} \tag{1.4}$$

and $\delta=\mu+J/2$. The last two terms in (1.3) correspond to a rewriting of the AF Heisenberg term plus a hard–core repulsion replacing Gutzwiller projection. [Summation over repeated spin indices is understood and dependence on euclidean time x^0 is not explicitly exhibited; $\beta=\frac{1}{k_BT}$, with k_B the Boltzmann constant].
For example, the grand canonical partition function of the $t-J$ model is expressed in path–integral form as

$$\Xi(\beta,\mu)=\int\mathcal{D}\Psi\mathcal{D}\Psi^*e^{-S(\Psi,\Psi^*)} \tag{1.5}$$

2. "Chern–Simons representation" of the t–J model

A key problem is to find a good Mean Field Approximation (MFA) to analyze the $t-J$ model.

A priori, we can consider many possibilities as starting point for a MFA from a general procedure valid in 2D: the "Chern–Simons bosonization". Let W_μ be a gauge field of gauge group G; define the (euclidean) Chern–Simons action by

$$S_{c.s.}(W) = \frac{1}{4\pi i}\int_0^\beta dx^0 \int d\vec{x}\epsilon^{\mu\nu\rho}\mathrm{tr}\,[W_\mu\partial_\nu W_\rho + \frac{2}{3}W_\mu W_\nu W_\rho](x_0,\vec{x}); \tag{2.1}$$

denote by $S(\psi,\psi^*,W)$ the action obtained from $S(\psi,\psi^*)$ by minimally coupling ψ,ψ^* to the gauge field W.

The output of "Chern–Simons bosonization" can be summarised as follows: In $2D$, for suitable choices [11] of the group G and of real coefficient(s) k_G, one can replace the path–integration over Ψ,Ψ^* by path–integration over W and new spin $\frac{1}{2}$ fields, χ,χ^*, bosonic or fermionic depending on $\{G,k_G\}$, substituting the action $S(\Psi,\Psi^*)$ by

$$S(\chi,\chi^*,W) + k_G S_{c.s.}(W) \tag{2.2}$$

(with suitable b.c. [11]) and the new theory is exactly equivalent to the original $t-J$ model.

Remarks

1) Although the C.S. bosonization is an exact identity if treated without approximations, each one of these "C.S. representations" in terms of χ,χ^*,W can be taken as a starting point for a different MFA. As an axample, if we choose $G=U(1), k_{U(1)}=1$ and χ,χ^* bosonic, one can reproduce [12] the "slave–boson" approach [13].

2) A similar strategy works for the Fractional Quantum Hall Effect: the system in a plateau of Hall conductivity around a filling fraction $\nu = \frac{1}{2\ell+1}, \ell$ integer, of the first Landau level appears [14] to have a good MFA in terms of a "C.S. bosonization" of the original long–distance action with $G=U(1)$, $k_{U(1)} = 2\ell+1$.

Here we sketch how the C.S. bosonization works intuitively, while more details are given in the Appendix (for a rather complete discussion see [11]). [In the following, we usually omit explicit reference to the conjugate fields, like Ψ^*].

Integrating out the time–component of the gauge field, W_0, appearing linearly in the action, one obtains a constraint of the form

$$j_0(x) = \frac{k_G}{2\pi} \epsilon_{0\nu\rho} W^{\nu\rho}(x), \tag{2.3}$$

where $W^{\mu\nu}$ is the field strength associated with W_μ (in particular W^{12} is the "G–magnetic field") and j_μ is the "G–current" of the matter field.

As a result a "G–vortex" is attached to every particle described by χ, χ^*, hence assigning to them a "G–magnetic charge". These particles also carry a "G–electric charge", since χ is minimally coupled to W_μ. The presence of both "electric" and "magnetic" charges implies a Aharonov–Bohm (A–B) effect when these particles are exchanged, thus introducing a "phase" for every exchange [15]. If this "A–B phase" is trivial (+1), then the statistics of χ is unchanged after W–integration and, to match with that of Ψ, the field χ must be taken fermionic. If the "A–B phase" is –1, then integration over W turns the statistics of a bosonic χ into fermionic, thus matching with that of Ψ (after W–integration).

Furthermore one can prove that the only effect of the C.S. term is to introduce the "A–B phase". These arguments give an idea why C.S. bosonization is an exact identity.

To analyze the $t-J$ model we choose χ fermionic,

$$G = U(1) \times SU(2), \quad k_{U(1)} = -2, \quad k_{SU(2)} = 1. \tag{2.4}$$

The gauge field with gauge group $U(1)$ is denoted by B_μ and the gauge field with gauge group $SU(2)$ is denoted by V_μ. Why we made this choice?

The basic argument is that with this choice, applying a dimensional reduction ($2D \to 1D$) and performing a MFA we are able [16] to reproduce the exact results on the one–dimensional $t-J$ model in the limit $t >> J$, obtained by Bethe–Ansatz and Conformal Field Theory techniques, including the "semionic" (i.e. intermediate between bosonic and fermionic [15]) statistics of spin and charge excitations.

3. A formal separation of charge and spin degrees of freedom

In $2D$ we separate formally [17] the spin and charge degrees of freedom (d.o.f.) of χ (the fermion of the $t-J$ model in the chosen "$U(1) \times SU(2)$ Chern–Simons representation") by a polar decomposition:

$$\chi_\alpha = H \Sigma_\alpha. \tag{3.1}$$

In (3.1), H is a spinless fermionic field ("holon") and it is minimally coupled to B respecting a local $U(1)$ gauge invariance; Σ_α is a spin $\frac{1}{2}$ bosonic field ("spinon"),

it is minimally coupled to V respecting a local $SU(2)$ gauge invariance and it satisfies the constraint

$$\Sigma^*_{\alpha j}\Sigma_{\alpha j}=1 \tag{3.2}$$

for each site j. An additional abelian local gauge invariance, called here h/s, appears due to the decomposition (3.1) of χ. It corresponds to the addition of a local phase to H and subtraction of the same phase from Σ_α.

<u>Remarks</u>

1) Fields with the same quantum numbers and constraints of those appearing in the "slave fermion" approach [18] are obtained from H,Σ_α by performing a Holstein–Primakoff transformation [12], but the action derived in the "$U(1)\times SU(2)$ C.S. representation" differs from the slave fermion one. At this stage it is explicitly given [17] by

$$S(H,\Sigma,B,V)=\int_0^\beta dx^0\{\sum_j[H^*_j(\partial_0-iB_0(j)-\delta)H_j+iB_0(j)$$

$$+(1-H^*_jH_j)\Sigma^*_{j\alpha}(\partial_0+iV_0(j))_{\alpha\beta}\Sigma_{j\beta}]+\sum_{<ij>}[(-tH^*_je^{i\int_{<ij>}B}H_i\Sigma^*_{i\alpha}(Pe^{i\int_{<ij>}V})_{\alpha\beta}\Sigma_{j\beta}+$$

$$+h.c.)+\frac{J}{2}(1-H^*_jH_j)(1-H^*_iH_i)(|\Sigma^*_{i\alpha}(Pe^{i\int_{<ij>}V})_{\alpha\beta}\Sigma_{j\beta}|^2-\frac{1}{2})]\}-2S_{c.s.}(B)+S_{c.s.}(V). \tag{3.3}$$

2) The fermionic field Ψ itself can be written as a product of two $U(1)\times SU(2)$–gauge invariant fields, one constructed out of H and B and one out of Σ_α and V; the statistics of both these fields is semionic, as originally suggested by Laughlin [19].

As usual in the path–integral formalism, to proceed we need a gauge–fixing of the local gauge symmetries of the action [20].

We gauge fix the $U(1)$ symmetry by imposing a Coulomb gauge on B. Integrating out the time component B_0, appearing linearly in the action, one obtains

$$B_\mu=\bar{B}_\mu+\delta B_\mu(H),\quad \mu=1,2, \tag{3.4}$$

where $\bar{B}_\mu$ is a mean field introducing a flux π per plaquette p, i.e. $e^{i\int_{\partial p}\bar{B}}=-1$, and δB_μ is a fluctuation term depending on holon density.

We gauge–fix the $SU(2)$ symmetry retaining the bipartite structure appearing in the ground state at zero doping, i.e. the Néel state of the AF Heisenberg model.

For this purpose, we adopt the "Néel gauge" defined by $\Sigma_{j\alpha} = \sigma_x^{|j|} \binom{1}{0}$, $|j| = j_1 + j_2$, where (j_1, j_2) denote the coordinates of the site j. The "spins" $\Sigma^* \vec{\sigma} \Sigma$ are then forced in the Néel configuration, leaving all the $SU(2)$ degrees of freedom in the gauge field V, unconstrained. We decompose V into a "Coulomb component" V^c satisfying

$$\partial^\mu V^c_\mu = 0 \quad \mu = 1, 2$$

and gauge fluctuations around it, described by an $SU(2)$ field g. Integrating out the time–component V_0, appearing linearly in the action, we obtain an explicit expression of V^c_μ in terms of g and H, denoted by $V_\mu(g, H)$.
Following a strategy developed in $1D$ [16], with techniques patterned from a proof of the diamagnetic inequality [21], we then find a g–configuration, g^m, optimizing the partition function of holons in a fixed g–background [17].
For small but non-vanishing doping concentration we find that $\bar{V}(H) \equiv V^c(g^m, H)$ attaches to the holons a vortex of $SU(2)$–vorticity $\pm \sigma_z \frac{\pi}{2}$, where σ_z is the diagonal Pauli matrix, and the sign depends on the Néel sublattice where the holon is located. Furthermore we have $g^m_j \sigma_x^{|j|} = \sigma_x^{|j|+1}$. We rewrite

$$V^c_\mu(g, H) = \bar{V}_\mu(H) + \delta V_\mu(g, H), \quad \mu = 1, 2, \tag{3.5}$$

where the fluctuation term δV depends also on the spin d.o.f. described by g, whereas $\bar{V}$ is independent of them.

Remark

At this stage, performing a simple field redefinition, the action of the $t - J$ model can be exactly written [17] in terms of H and g as $S = S_h + S_s$, where

$$S_h = \int_0^\beta dx^0 \{ \sum_j H^*_j (\partial_0 - (\sigma_x^{|i|} g^\dagger_j \partial_0 g_j \sigma_x^{|j|})_{11} - \delta) H_j$$

$$+ \sum_{<ij>} [-t H^*_j e^{-i \int_{<ij>} \bar{B} + \delta B} H_i (\sigma_x^{|i|} g_i P(e^{i \int_{<ij>} \bar{V} + \delta V}) g_j \sigma_x^{|i|})_{11} + h.c.] \}$$

$$S_s = \int_0^\beta dx^0 \{ \sum_j (\sigma_x^{|j|} g^\dagger_j \partial_0 g_j \sigma_x^{|j|})_{11} +$$

$$+ \sum_{<ij>} \frac{J}{2} (1 - H^*_i H_i)(1 - H^*_j H_j) \big[|(\sigma_x^{|i|} g^\dagger_i P(e^{i \int_{<ij>} \bar{V} + \delta V}) g_j \sigma_x^{|j|})_{11}|^2 - \frac{1}{2} \big] \} \tag{3.6}$$

We now make the first basic approximation neglecting δB_μ and δV_μ, i.e. the feed–back of charge fluctuations on B and of spin fluctuations on V^c, still retaining the holon dependence of V^c. Presumably the main neglected effect is a statistic transmutation giving rise to semionic statistics for holons and spinons, as in $1D$. We argue that the statistics here is less relevant than in $1D$ because we expect in $2D$ the formation of a bound state with the quantum numbers of the electron, due to gauge fluctuations discussed later on.

4. Low energy effective action for spin degrees of freedom

To derive a low–energy effective action for the spin d.o.f., we first rewrite g in CP^1 form, introducing a spin $\frac{1}{2}$ field b_α through

$$g_j = \begin{pmatrix} b_{1j} & -b^*_{2j} \\ b_{2j} & b^*_{1j} \end{pmatrix} \tag{4.1}$$

with the constraint $b^*_{j\alpha} b_{j\alpha} = 1$ for every site j.
Subsequently, we apply to b_α the standard treatment of AF systems splitting it into an AF component, described by a spin $\frac{1}{2}$ complex boson field $z_\alpha, \alpha = 1, 2$ and a ferromagnetic component which is then integrated out [22].
The continuum action is then given in CP^1 form by

$$S(z, A) = \int_0^\beta dx^0 \int d\vec{x} \{ |(\partial_0 - A_0) z_\alpha|^2 + v_s^2 |(\partial_\mu - A_\mu) z_\alpha|^2 + \bar{V}^2(H) z^*_\alpha z_\alpha \}(x^0 \vec{x}), \tag{4.2}$$

where v_s is the "spin velocity", J-dependent, and A is the standard "Hubbard–Stratonovich" gauge field of the CP^1 models. [In the derivation we implicitly assumed that the CP^1 model is in the symmetric phase; this is self–consistent with the behaviour discussed later on].
This action describes spin waves $\vec{\Omega} = z^* \vec{\sigma} z$ interacting with the vortices appearing in $\bar{V}^2$, centered at the holon positions. We evaluate approximately the effect of $\bar{V}^2$ averaged over holon positions at fixed density δ. One can argue that this treatment can be justified for $J << t$, because the holon appears to develop a large effective mass, due to the coupling to soft spin fluctuations [23].
The output is an average

$$\langle \bar{V}^2 \rangle \sim -\delta \ln \delta. \tag{4.3}$$

Substituting in (4.2) $\bar{V}^2$ by $< \bar{V}^2 >$ produces a mass for the spin waves, suggesting that the system should exhibit short–range AF, with a correlation length $\xi_{AF} \sim$

$(-\delta \ln \delta)^{-\frac{1}{2}}(\sim \delta^{-\frac{1}{2}}$ for δ not too small). This result is in agreement with data deduced from neutron experiments in $La_{2-\delta}Sr_\delta CuO_4$, where a fit for ξ_{AF} was suggested [24] in terms of $\delta^{-\frac{1}{2}}$, as shown in fig. 3.

Remark

If $S(z, A)$ would be the full action, the spinons z would be logarithmically confined by the Coulomb interaction mediated by A, due to the massive nature of z, but coupling with holons yields deconfinement.

5. Low energy effective action for charge degrees of freedom

Introducing a flux π per plaquette, the mean field $\bar{B}$ induces a partition of the lattice in two Néel sublattices (here denoted $\uparrow$ and $\downarrow$) and it converts the spinless fermion H into 2–components Dirac–like fermions of two–species:

$$\psi^{(1)} = \begin{pmatrix} \psi_\uparrow^{(i)} \\ \psi_\downarrow^{(1)} \end{pmatrix} \qquad \psi^{(2)} = \begin{pmatrix} \psi_\downarrow^{(2)} \\ \psi_\uparrow^{(2)} \end{pmatrix}, \tag{5.1}$$

each component of them being supported on a Néel sublattice, as indicated by the subscript. The vertices of the cones of the "Dirac" energy–momentuum dispersion relation are centered at the four points $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$ in the Brillouin zone. (This phenomenon is standard in the flux phase; see [22,25]). Up to a short–range term, the continuum action for these fermions is given by

$$S(\psi, A) = \int_0^\beta dx^0 \int d\vec{x} \{ \sum_{r=1}^{2} \bar{\psi}^{(r)} [\gamma_0(\partial_0 - \delta - e^{(r)} A_0) - \tilde{t}(\not{\partial} - e^{(r)} \not{A})] \psi^{(r)} \}(x^0, \vec{x}), \tag{5.2}$$

where $\tilde{t}$ is a renormalized hopping parameter and $e^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e^{(2)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.
[We adopt the standard notation

$$\gamma_0 = \sigma_z \quad \gamma_\mu = (\sigma_y, \sigma_x) \quad \not{A} = \gamma_\mu A_\mu \quad \bar{\psi} = \psi^* \gamma_0 \quad].$$

The components of the fermion fields in (5.2) supported in the two Néel sublattices have opposite charge with respect to A, which is the same gauge field appearing in (4.2). It can be traced back as the gauge field of h/s gauge invariance.
With respect to the action discussed in [25], a crucial difference is the appearance of the $\gamma_0\delta$ term. Neglecting at first A, this term produces a finite Fermi surface for the gapless components $(\psi_\uparrow^{(1)}, \psi_\downarrow^{(2)})$. The other two components have a gap and

they mix with the gapless ones, due to the presence of the (non–diagonal) Dirac γ matrices.
This mixing is expected to produce a reduction of the spectral weight in the outer part in the reduced Brillouin zone scheme.
In a similar lattice model (with a twist of statistics between holon and spinons) the shape of the Fermi surface for the "electron" deduced in mean fields is half–pocket like [26], showing a qualitative agreement with the F.S. deduced from ARPES experiments in underdoped cuprates [6] (see fig. 4).

6. Final comments

We can summarize the large distance behaviour of the model in $U(1)\times SU(2)$ C.S. representation (with $\delta B = \delta V = 0$), in terms of a system of spinons, z_α, whose dynamics is described by a CP^1 model with mass term $m \sim \delta^{\frac{1}{2}}$ (the main novelty) and a system of Fermi liquid holons (obtained by integrating out the gapful Dirac modes) with $\epsilon_F \sim t\delta$, interacting via an abelian gauge field A.
The $U(1)$ effective action for A then exhibits the basic features, such as the Reizer [27] singularity in the propagator of the transverse component of A and the existence of a characteristic energy scale for spinons, that permits one to apply the ideas of [28] to deduce a linear in T in–plane resistivity for certain temperature range.
A more careful study of in–plane resistivity in the present model is in prgress [29].
The analysis presented in this paper applies only to small doping concentration δ, since we derived under this condition (see eq. 3.5–4.3) the form of $\bar{V}$ used in the treatment, and crucial to get a mass term for spinons. Hence the above description is applicable to underdoped materials, as explicitly stated in Sec.1. The extension to other regions of the phase diagram is an open problem for further investigation.

Appendix: Chern–Simons bosonization

We start by rewriting the grand canonical partition function $\Xi(\beta,\mu)$ of the $t-J$ model in terms of the canonical partition function Z_N at fixed number of fermions N:

$$\Xi(\beta,\mu) = \sum_N \frac{e^{\beta\mu N}}{N!} Z_N(\beta). \qquad (A.1)$$

Following a standard treatment of Feynman path–integral in the continuum [10], adapted to the lattice, we can express Z_N as a sum over virtual trajectories of

fermions on a lattice with imaginary time. These trajectories start at the imaginary time $x^0 = 0$ at a set of lattice sites $\{j_1, ..., j_N\}$ and they end at the imaginary time $x^0 = \beta$ at the same set of sites permuted, $\{j_{\sigma(1)}, ..., j_{\sigma(N)}\}$, where σ is a permutation (see fig.5). Each term in the sum has a weighting factor $(-1)^{\epsilon(\sigma)}$, where $\epsilon(\sigma)$ is the number of exchanges in σ. By omitting this factor one reproduces the canonical partition function, Z_N^b, of a fictitious "bosonic $t-J$ model" obtained by substituting fermions with bosons in the $t-J$ model. The hard–core constraint forbids any intersection among the trajectories of the N fermions and, by periodicity in imaginary time (the planes $x^0 = 0$ and $x^0 = \beta$ are identified, see [10]), every set of virtual trajectories defines a link $\mathcal{L}$ (see fig.5). The factor $(-1)^{\epsilon(\sigma)}$ is a topological invariant associated to that link, i.e. it does not change under an arbitrary deformation of the link performed without allowing intersections. According to a general theory [30] every topological invariant of links can be represented as the expectation value of a "Wilson loop" supported on the link, in a gauge theory with Chern–Simons action $S_{c.s.}$.

More precisely, let W_μ denote a gauge field of gauge group G and k_G a real coefficient. Then, for a suitable choice of G and k_G (for explicit conditions, see [11]), we have

$$(-1)^{\epsilon(\sigma)} = \int \mathcal{D}W e^{-k_G S_{c.s.}(W)} \mathrm{Tr}\; P(e^{i\int_{\mathcal{L}} W_\mu dx^\mu}), \qquad (A.2)$$

where $\mathcal{L}$ is a link associated with a set of virtual trajectories whose end points are obtained by a permutation σ from the initial points. In (A.2) the last factor is the "Wilson loop", where Tr denotes the normalised trace and $P(\cdot)$ denotes the normal ordering, which amounts to the usual time ordering $T(\cdot)$ for a fictitious "time" parametrising the link. Furthermore, the normalization is chosen such that $\int \mathcal{D}W \exp\{-k_G S_{c.s.}(W)\} = 1$.

Let us denote by $Z_N^b(W)$ the canonical partition function of the modified "bosonic $t-J$ model" with the boson field minimally coupled to the gauge field W. It is easy to verify (see e.g.[30] for the abelian and [12] for the non-abelian case) that the dependence of W in $Z_N^b(W)$ appears in the form of linear combinations of Wilson loops on the links associated with virtual trajectories. Therefore the identity $(A.2)$ implies that

$$Z_N = \int \mathcal{D}W e^{-k_G S_{c.s.}(W)} Z_N^b(W). \qquad (A.3)$$

Equation (A.3) is a "bosonization formula" and it holds for all the couples of G and k_G satisfying (A.2). If we plug (A.3) into (A.1), we obtain

$$\Xi = \int \mathcal{D}W e^{-k_G S_{c.s.}(W)} \Xi^b(W), \tag{A.4}$$

where $\Xi^b(W)$ is the grand–canonical partition corresponding to $Z_N^b(W)$, given by

$$\Xi^b(W) = \int \mathcal{D}\chi \mathcal{D}\chi^* e^{-k_G S(\chi,\chi^*,W)}, \tag{A.5}$$

with χ a bosonic field. Combining equations (A.4) and (A.5), one obtains the "Chern–Simons bosonization" discussed in the text. It is obvious that introducing further Chern–Simons gauge fields we can turn χ into a fermion, following a similar procedure.

Remarks

1) A warning: in the above discussion subtle points like boundary condition and "framing" are completely ignored, see [11,30].

2) To have a more concrete feeling about formula (A.2), let us consider the simple case $G = U(1), k_{U(1)} = 1$, a choice for which the formula holds, and, due to the abelian nature of G, path–ordering is not needed, and the trace Tr is trivial.

Let us denote by $\Sigma_\mu dx^\mu$ a "singular δ–like current" supported on a surface Σ, whose boundary is given by the link $\mathcal{L}$. Then one easily verifies that

$$e^{i\int_{\mathcal{L}} W_\mu dx^\mu} = e^{i\int W_\mu \epsilon^{\mu\nu\rho}\partial_\nu \Sigma_\rho d^3x}. \tag{A.6}$$

We now compute

$$\int \mathcal{D}W e^{\frac{i}{4\pi}\int \epsilon^{\mu\nu\rho} W_\mu \partial_\nu W_\rho d^3x} e^{i\int W_\mu \epsilon^{\mu\nu\rho}\partial_\nu \Sigma_\rho d^3x} \tag{A.7}$$

by a change of variable. Shifting $W_\mu \to W_\mu + 2\pi\Sigma_\mu$ and "completing the square", one obtains

$$\int \mathcal{D}W e^{\frac{i}{4\pi}\int \epsilon^{\mu\nu\rho} W_\mu \partial_\nu W_\rho d^3x} e^{-\frac{i}{2\pi}\frac{1}{2}(2\pi)^2 \int \epsilon^{\mu\nu\rho}\Sigma_\mu \partial_\nu \Sigma_\rho d^3x}$$

$$= e^{-i\pi \int \Sigma_\mu \epsilon^{\mu\nu\rho}\partial_\nu \sum_\rho d^3x} = e^{-i\pi \int_{\mathcal{L}} \Sigma_\mu dx^\mu}. \tag{A.8}$$

The integral in the exponent of the last term in (A.8) receives a contribution only when $\mathcal{L}$ crosses the surface Σ, i.e. for every crossing appearing in $\mathcal{L}$, i.e. for every exchange of the particles in the virtual trajectories, so that its value is given by $(-1)^{\epsilon(\sigma)}$, q.e.d. .

Acknowledgments. I wish to thank J. Fröhlich, Z-B. Su and L. Yu for the joy of collaboration. Useful discussions with N.P. Ong and F.Toigo are also gratefully acknowledged.

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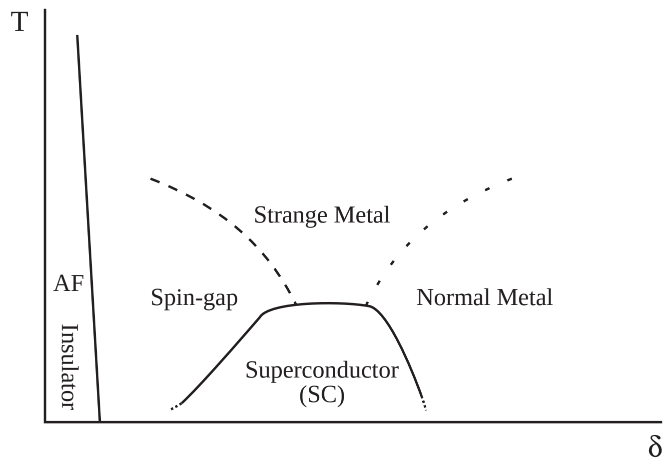


Fig. 1. Presumed phase diagram.

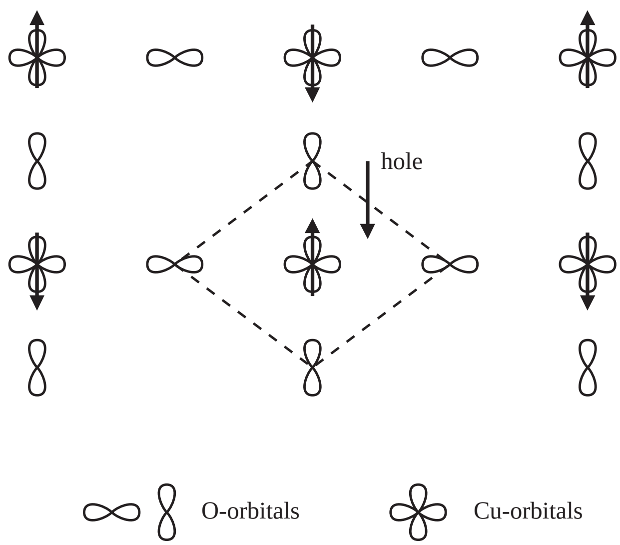


Fig. 2. Zhang-Rice singlet.

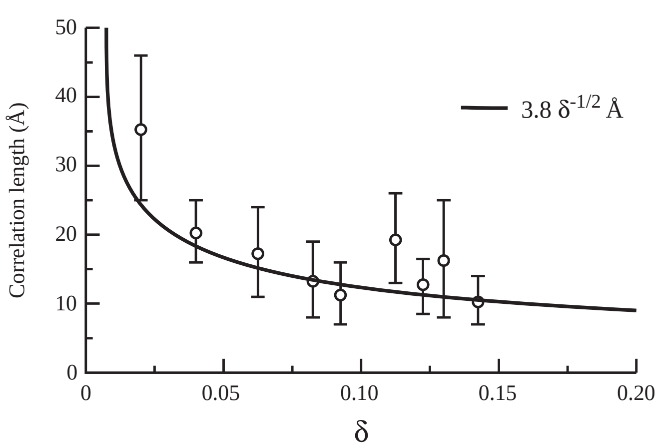
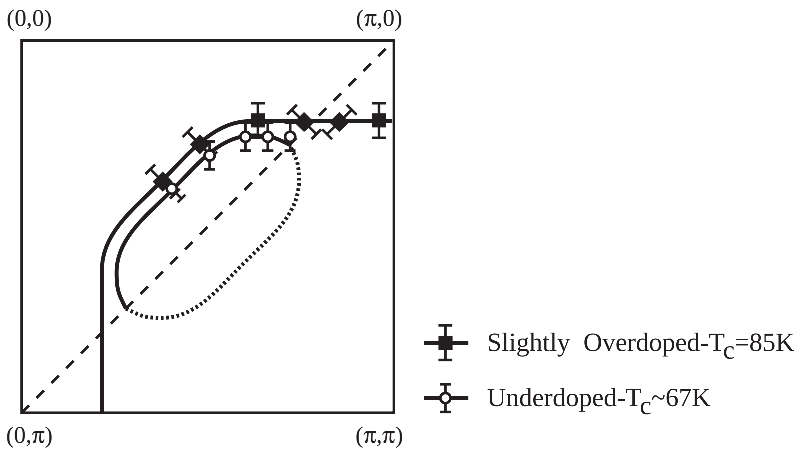


Fig. 3. Instantaneous spin correlation length vs temperature in $La_{2-\delta}Sr_{\delta}CuO_4$.



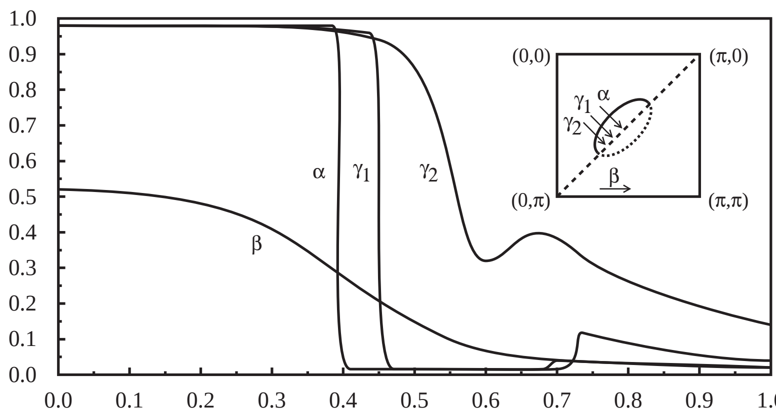


Fig. 4. Fermi level crossing from two Bi2212 samples of differing oxygen content (up) compared with the momentum distribution of “electrons” found in [26] (down).

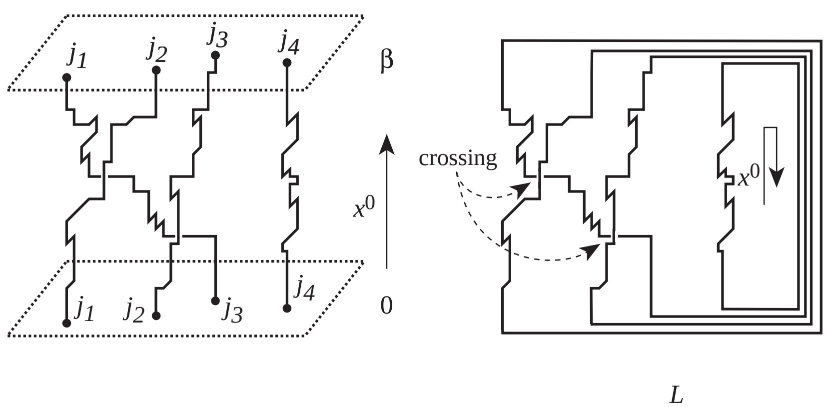


Fig. 5. Heavy lines describe a set of virtual trajectories for N=4; L is the link obtained identifying the x^0=β and x^0=0 planes.